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Cyclic representation and function difference representation of the Z_n Sklyanin algebra

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Abstract. We obtain the cyclic representation of Z_n Sklyanin algebra. From this we derive its function difference representation. When $n=2$, it coincides with the known result.

1. Introduction

Recently, there has been considerable attention and intensive study on quantum group theory. From the physics point of view, the quantum group can represent both the exchange relation symmetry of vertex operators in conformal field theory and the symmetry of the six-vertex model and other exactly solvable statistical models, which have triangular functions as their Boltzmann weights.

It is expected that Sklyanin algebra [1] plays a similar role in integrable-massive field theory [2], which has its exchange relations expressed by elliptic functions, and in some exactly solvable statistical models [3–8], which have their Boltzmann weights expressed in elliptic functions. They are reduced to the corresponding triangular models when the modular parameter approaches infinity.

Starting from the eight-vertex model [3], Sklyanin [7] derived the sl(n) Sklyanin algebra for $n=2$. He subsequently constructed the single variable function representation of the algebra, and consequently the classification of the representations. Using the limit of these representations, he was the first to give the highest weight representation and the minimal cyclic representation of the quantum group $\mathrm{sl}_q(2)$.

Starting from $n\otimes n$ [3, 9] model, Cherednik [10] generalized the Sklyanin algebra to the generic sl(n). Wei *et al* [11, 12] worked out the structure constants for the algebra. The authors of this paper gave explicitly [13] the process by which Sklyanin algebra is reduced to $U_q(\tilde{\mathrm{sl}}(n))$. Zhou *et al* [14, 15] constructed the tensor product representation of the Sklyanin algebra through fusion. However, the explicit expressions of the cyclic representation have yet to be found.

Hasegawa and Yamada [16] managed to construct the Yang–Baxter operator $L(u)$ for the eight-vertex model by using the cyclic representation [1] of Sklyanin algebra. It was found that the operator $L(u)$ could be factorized, which is similar to the triangular case in [18, 19]. They further derived the broken Z_n model [7, 8]. Bazhanov *et al* [19] point out that the factorizability of $L(u)$ means that we could construct the IRF model from the corresponding vertex model. Recently, making use of this approach, Quano

[20] worked out the cyclic representation of the $L(u)$ operator for the Sklyanin algebra. (He also gave a new type of $A_n^{(1)}$ Kashiwara–Miwa model [6, 7].)

In this paper, we give an explicit cyclic representation of the Sklyanin operator S_α, and we find that it has $2n-1$ parameters, which is the same number as for the minimal cyclic representation of quantum group $U_q(\mathrm{sl}(n))$. In the mean time, the number of parameters for the cyclic representation of the $L(u)$ operator is also increased. On the other hand, the difference expression of the Sklyanin algebra operator, which is highly lauded by Smith [22], is the foundation [1] of representation theory for the $n=2$ case. We can make further study of this expression and compare it with its triangular limit, the quantum algebra.

1.1. The intertwiner for $A_{n-1}^{(1)}$ IRF model and Z_n symmetric vertex model and its factorized YBE operator $L(z)$

1.1.1. The Z_n symmetric Belavin R-matrix and Sklyanin algebra (SA). For a given positive integer n, we define $n\otimes n$ matrices g, h, I_α:

$$g_{jk}=\omega^j\delta_{jk} \qquad h_{jk}=\delta_{j+1,k} \qquad \omega=\exp\left(\frac{2\pi \mathrm{i}}{n}\right) \tag{1}$$
$$I_\alpha=I_{(\alpha_1,\alpha_2)}=g^{\alpha_2}h^{\alpha_1}.$$

Let $I_\alpha^{(j)}=I\otimes\ldots\otimes I_\alpha\otimes\ldots$, I_α is at the jth space

$$W_\alpha(z)=\theta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_2}{n}\\ \frac{1}{2}+\frac{\alpha_1}{n}\end{bmatrix}\left(z+\frac{w}{n},\tau\right)\Bigg/\theta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_2}{n}\\ \frac{1}{2}+\frac{\alpha_1}{n}\end{bmatrix}\left(\frac{w}{n},\tau\right)\equiv\frac{\sigma_\alpha\left(z+\frac{w}{n}\right)}{\sigma_\alpha\left(\frac{w}{n}\right)} \tag{2}$$

$$\theta\begin{bmatrix}a\\ b\end{bmatrix}(z,\tau)=\sum_{m\in Z}\exp\{\mathrm{i}\pi(m+a)[(m+a)\tau+2(z+b)]\}$$

$$\sigma_\alpha(z)\equiv\theta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_2}{n}\\ \frac{1}{2}+\frac{\alpha_1}{n}\end{bmatrix}(z,\tau)$$

then the Z_n symmetric Belavin R-matrix is written as

$$R_{jk}(z)=\sum_{\alpha\in Z_n^2}W_\alpha(z)I_\alpha^{(j)}(I_\alpha^{-1})^{(k)}. \tag{3}$$

They satisfy the YBE

$$R_{12}(z_1-z_2)R_{13}(z_1)R_{23}(z_2)=R_{23}(z_2)R_{13}(z_1)R_{12}(z_1-z_2).$$

The operator representation of YBE is the $n\otimes n$ matrix $L_j(z)$ satisfying the following equation:

$$R_{12}(z_1-z_2)L_1(z_1)L_2(z_2)=L_2(z_2)L_1(z_1)R_{12}(z_1-z_2) \tag{4}$$

where $L_1(z_1)=L(z_1)\otimes I$, $L_2(z_1)=I\otimes L(z_2)$.

If the $L_j(z)$ could be expressed as

$$L_j(z) = \sum_{\alpha \in Z_n^2} W_\alpha(z+c) I_\alpha^{(j)} S_\alpha \tag{5}$$

then the operator S_α satisfies Z_n SA

$$\sum_\gamma C_{\alpha\beta\gamma}(w, \tau) S_{\alpha-\gamma} S_\gamma = 0 \tag{6}$$

where α, β, $\gamma \in Z_n^2$, structure constant $C_{\alpha\beta\gamma}$, operator S_α is independent of the spectrum parameter z. On the other hand, we can find a solution $L_j(z)$ of (4), if we have a representation of S_α satisfying (6),

1.1.2. Cyclic representation state and the Boltzmann weight of the IRF model. According to Jimbo *et al* [6], we have

$$W = \bigotimes_{j=0}^{n-1} V_j \qquad V_j \cong C^N.$$

We choose a set of canonical bases such that

$$Zu_j = u_{j-1} \qquad Xu_j = q^j u_j \qquad q = \exp\left(\frac{2\pi \mathrm{i}}{N}\right)$$

then we have the cyclic representation space W^0

$$W^0 = \{w \in W \mid Z_0 \ldots Z_{n-1} w = w\}.$$

This could be constructed from the base vector

$$w_m = \sum_{k=0}^{N-1} u_{m_0+k} \otimes \ldots \otimes u_{m_{n-1}+k} \qquad m = (m_0 \ldots m_{n-1}) \in Z_N^n \tag{7a}$$

obviously

$$w_{(m_0,\ldots,m_{n-1})} = w_{(m_0+k,\ldots,m_{n-1}+k)}$$

hence w_m can be uniquely described by the element in $\{Q\}$

$$Q = Z_N^n \bmod Z_N(1, \ldots, 1). \tag{7b}$$

We can choose from the equivalent $m \in Z_N^n$ one with $m_0 = 0$

$$\{0, m_1', m_2' \ldots m_{n-1}'\}$$

where $m_j' = m_j - m_0 \bmod N$.

Define

$$[m] = (0, \ldots, m_{n-1}') \tag{8}$$

then a base vector is determined by the $n-1$ numbers $m_j' \in Z_N$. We may obtain $[a]$ from a given vector $a \in \{w_m\}$; the opposite is also true, so that W^0 is N^{n-1} dimensional.

Given any two states $a, b \in \{w_m\}$, they are called admissible if they satisfy

$$[a] - [b] = [e_j] \bmod Z_N^n \qquad \text{with } 0 \leqslant j \leqslant n-1 \tag{9}$$

where $e_j = (0, \ldots, 1, 0 \ldots)$, 1 is at the jth place.

We write $b=a-e_j$, for a given state a there are n states b satisfying this relation. Define

$$W_z\begin{bmatrix} m & n-e_j \\ m-e_j & m-2e_j \end{bmatrix}=\frac{\sigma_0\left(z+\frac{N'}{N}\right)}{\sigma_0\left(\frac{N'}{N}\right)}$$

$$W_z\begin{bmatrix} m & m-e_j \\ m-e_j & m-e_j-e_k \end{bmatrix}=\frac{\sigma_0\left(z+m_{jk}\frac{N'}{N}\right)}{\sigma_0\left(m_{jk}\frac{N'}{N}\right)} \tag{10}$$

$$W_z\begin{bmatrix} m & m-e_j \\ m-e_k & m-e_j-e_k \end{bmatrix}=\frac{\sigma_0(z)\sigma_0\left(m_{jk}\frac{N'}{N}-\frac{N'}{N}\right)}{\sigma_0\left(\frac{N'}{N}\right)\sigma_0\left(m_{jk}\frac{N'}{N}\right)}$$

where m, e_j, $e_k \in Z_N^n$, N, N' coprime to each other

$$m_{jk}=\bar{m}_j-\bar{m}_k \qquad \bar{m}_j=m_j-\frac{1}{n}\sum_{j'=0}^{n-1} m_{j'}+w_j$$
$$w_j\neq w_k \bmod N\wedge \qquad \text{if } j\neq k. \tag{11}$$

We may also define Boltzmann weight

$$W^{\text{cyc}}\begin{bmatrix} a & b \\ d & c \end{bmatrix}$$

for a state configuration

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

round a face by $(a, b, c, d \in \{w_m\})$

$$W_z^{\text{cyc}}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{cases} W_z\begin{bmatrix} [a] & [b] \\ [c] & [d] \end{bmatrix} & \text{if } (a, b), (b, c), (a, d) \text{ and } (d, c) \text{ are admissible} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

It can be shown that the non-vanishing W_z^{cyc} is one of these in (10), furthermore it is single-valued with respect to Z_N^n (the right-hand side of (10) is invariant under $m_j \to m_j+N$).

1.1.3. The intertwiner of face-vertex models and factorized L(z). Jimbo *et al* [21, 20] introduced the following intertwiner

$$\varphi_{ml}(z) = {}^t(\varphi_{ml}^{(0)}(z), \ldots, \varphi_{ml}^{(n-1)}(z))$$

$$\varphi_{ml}^{(k)}(z) = \begin{cases} \theta^{(k)}(z + nw\bar{m}_j) & \text{if } [m]-[l]=e_j \\ 0 & \text{otherwise} \end{cases} \tag{13a}$$

$$\theta^{(j)}(u) \equiv \theta\begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{bmatrix}(u, n\tau).$$

This is a single-valued (with respect to m, l) n-dimensional vector. For a given state m, there are n states such that φ is non-vanishing. If different sets of numbers $(m_0, \ldots) \in Z_N^n$, $(m_0' \ldots) \in Z_N^n$ represent the same state, then (13) gives an identical vector $\varphi_m(z)$.

Jimbo *et al* [21] showed that, if $w = N'/N$, the following is true:

$$R(z^1 - z_2)\varphi_{ab}(z_1) \otimes \varphi_{bc}(z_2) = \sum_d W^{\mathrm{cyc}}_{z_1 - z_2}\begin{bmatrix} a & b \\ d & c \end{bmatrix} \varphi_{dc}(z_1) \otimes \varphi_{ab}(z_2). \tag{13b}$$

According to Bazhanov *et al* [17], if we could find a row vector

$$\bar{\varphi}_{ab}(z_1)(z) = (\bar{\varphi}_{ab}^{(0)}(z_1), \ldots, \bar{\varphi}_{ab}^{(n-1)}(z_1)$$

such that

$$\sum_k \bar{\varphi}_{ab}^{(k)} \varphi_{ac}^{(k)} = \begin{cases} \delta_{bc} & (a, b), (a, c) \quad \text{admissible} \\ 0 & \text{otherwise.} \end{cases} \tag{14a}$$

Consequently we have n b, c^s, furthermore, we have

$$\sum_b \varphi_{ab}^{(j)} \bar{\varphi}_{ab}^{(k)} = \delta_{jk} \tag{14b}$$

$$\bar{\varphi}_{dc}(z_1) \otimes \varphi_{ad}(z_2) R(z_1 - z_2) = \sum_b W^{\mathrm{cyc}}_{z_1 - z_2}\begin{bmatrix} a & b \\ d & c \end{bmatrix} \bar{\varphi}_{ab}(z_1) \otimes \bar{\varphi}_{bc}(z_2). \tag{15}$$

Proof. From (13)

$$R(z_1 - z_2)\varphi_{ab}(z_1) \otimes \varphi_{bc}(z_2) = \sum_d W_{z_1 - z_2}\begin{bmatrix} a & b \\ d & c \end{bmatrix} \varphi_{dc}(z_1) \otimes \varphi_{ab}(z_2).$$

Multiplying both sides from the right by $\bar{\varphi}_{ab}(z_1) \otimes \bar{\varphi}_{bc}(z_2)$ and summing over b, c we have

$$\begin{aligned} &\sum_b \sum_c R(z_1 - z_2)\varphi_{ab}(z_1)\bar{\varphi}_{ab}(z_1) \otimes \varphi_{bc}(z_2)\bar{\varphi}_{bc}(z_2) \\ &\quad = \sum_b R(z_1 - z_2)\varphi_{ab}(z_1)\bar{\varphi}_{ab}(z_1) \otimes I \\ &\quad = R(z_1 - z_2) I \otimes I \\ &\quad = \sum_{bc} \sum_d W_{z_1 - z_2}\begin{bmatrix} a & b \\ d & c \end{bmatrix} \varphi_{dc}(z_1)\bar{\varphi}_{ab}(z_1) \otimes \varphi_{ad}(z_2)\bar{\varphi}_{bc}(z_2). \end{aligned}$$

Then we multiply both sides from the left by $\bar{\varphi}_{d'c'}(z_1)\otimes\bar{\varphi}_{ad'}(z_2)$ to give

$$\begin{aligned}
&\bar{\varphi}_{d'c'}(z_1)\otimes\bar{\varphi}_{ad'}(z_2)R(z_1-z_2)\\
&\qquad=\sum_{bc}\sum_{d} W_{z_1-z_2}\begin{bmatrix} a & b\\ d & c\end{bmatrix}\bar{\varphi}_{d'c'}(z_1)\varphi_{dc}(z_1)\bar{\varphi}_{ab}(z_1)\otimes\delta_{dd'}\bar{\varphi}_{bc}(z_2)\\
&\qquad=\sum_{bc} W_{z_1-z_2}\begin{bmatrix} a & b\\ d' & c\end{bmatrix}\delta_{c'c}\bar{\varphi}_{ab}(z_1)\otimes\bar{\varphi}_{bc}(z_2)\\
&\qquad=\sum_{b} W_{z_1-z_2}\begin{bmatrix} a & b\\ d' & c'\end{bmatrix}\bar{\varphi}_{ab}(z_1)\otimes\bar{\varphi}_{bc'}(z_2).
\end{aligned}$$

Thus we have

$$\bar{\varphi}_{dc}(z_1)\otimes\bar{\varphi}_{ad}(z_2)R(z_1-z_2)=\sum_{b} W_{z_1-z_2}\begin{bmatrix} a & b\\ d & c\end{bmatrix}\bar{\varphi}_{ab}(z_1)\otimes\bar{\varphi}_{bc}(z_2).$$

Let

$$[L(z)]_{ab}=\varphi_{ab}(z)\bar{\varphi}_{ab}(z) \tag{16}$$

then $L(z)$ satisfies (5).

Proof. As

$$[R(z_1-z_2)L_1(z)L_1(z)]_{ac}=\sum_{b} R(z_1-z_2)\varphi_{ab}(z_1)\bar{\varphi}_{ab}(z_1)\otimes\varphi_{bc}(z_2)\bar{\varphi}_{bc}(z_2) \tag{16a}$$

from (14), the right-hand side of (16*a*) is

$$\begin{aligned}
&\sum_{b}\sum_{d} W_{z_1-z_2}\begin{bmatrix} a & b\\ d & c\end{bmatrix}\varphi_{dc}(z_1)\bar{\varphi}_{ab}(z_1)\otimes\varphi_{ad}(z_2)\bar{\varphi}_{bc}(z_2)\\
&\qquad=\sum_{d}\varphi_{dc}(z_1)\otimes\varphi_{ad}(z_2)\sum_{b} W_{z_1-z_2}\begin{bmatrix} a & b\\ d & c\end{bmatrix}\bar{\varphi}_{ab}(z_1)\bar{\varphi}_{bc}(z_2).
\end{aligned}$$

From (15), the left-hand side of (16*a*) is

$$\begin{aligned}
&\sum_{d}\varphi_{dc}(z_1)\otimes\varphi_{ad}(z_2)\bar{\varphi}_{dc}(z_1)\otimes\bar{\varphi}_{ad}(z_2)R(z_1-z_2)\\
&\qquad=\sum_{d}\varphi_{dc}(z_1)\bar{\varphi}_{dc}(z_1)\otimes\varphi_{ad}(z_2)\bar{\varphi}_{ad}(z_2)R(z_1-z_2)\\
&\qquad=[L_2(z_2)L_1(z_1)R(z_1-z_2)]_{ac}.
\end{aligned}$$

□

This is the cyclic $L(z)$ obtained by Quano [20]. It generalizes the results of Hasegawa *et al* [16].

Remark. We can generalize the $L(z)$ in (16) as follows:
Put $\varphi_{ab}(z+\zeta)$ instead of $\varphi_{ab}(z)$ in (16), then the resulting

$$[L(z)]_{ml}=\varphi_{ml}(z+\zeta)\bar{\varphi}_{ml}(z) \tag{16b}$$

also satisfies YBE (5).

The $\bar{\varphi}^{(i)}_{a,a-e_i}(z)$ which satisfies (14*a*) is the matrix element of the inverse of $N\otimes n$ matrix A, where $A_{ij}=\varphi^{(i)}_{a,a-e_i}$, i.e.

$$A=\begin{bmatrix} \theta^{(0)}(nz_0) & \dots & \theta^{(0)}(nz_{n-1}) \\ \vdots & \dots & \vdots \\ \theta^{(n-1)}(nz_0) & \dots & \theta^{(n-1)}(nz_{n-1}) \end{bmatrix} \tag{17}$$

where $nz_j=z+nw\bar{m}_j$. Thus

$$\bar{\varphi}^{(i)}_{a,a-e_i}(z)=B_{ij}(z)/\mathrm{Det}\ A \tag{18}$$

where B_{ij} is the cofactor matrix of A_{ij}. It can be shown [20] that

$$\mathrm{Det}\ A=C\sigma_0\left(\sum_{j=0}^{n-1} z_j-p_n\right)\prod_{l<k}\sigma_0(z_i-z_k) \tag{19}$$

where C is a z_i-independent constant and $p_n=(n-1)/2$.

2. Cyclic and function difference representations of SA

2.1. Cyclic representation of SA

We now look for the S_α in (6) which corresponds to $L(z)$ in (16). As I_α is invertible, and

$$\mathrm{tr}\ I_\alpha(I_\beta)^{-1}=n\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} \tag{20}$$

it is easy to show that $\{I_\alpha\}$ forms a complete set of bases. Let

$$L(z)=\sum_\alpha I_\alpha U_\alpha(z) \qquad U_\alpha:\ \text{operator}$$

then

$$U_\alpha(z)=\frac{1}{n}\ \mathrm{tr}\ L(z)(I_\alpha)^{-1}.$$

If $U_\alpha(z)=W_\alpha(z+\mu, w, \tau)S_\alpha$, where W_α is given by (2), then S_α is a representation of SA and satisfies (7). From (16)

$$L_{ml}(z)=\varphi_{ml}\bar{\varphi}_{ml}(z+\zeta) \tag{21}$$

$$\begin{aligned} n[U^\alpha]_{ml} &= \mathrm{tr}[L_{ml}(z)(I_\alpha)^{-1}] \\ &= \sum_{ik}[\varphi^{(i)}_{ml}(z)][\bar{\varphi}^{(k)}_{ml}](I_\alpha)^{-1}_{kl} \\ &= \sum_{ik}\bar{\varphi}^{(k)}_{ml}(z)(I_\alpha)^{-1}_{ki}\varphi^{(i)}_{ml}(z+\zeta) \\ &= \bar{\varphi}_{ml}(z)(I_\alpha)^{-1}\varphi_{ml}(z+\zeta) \\ &= \begin{cases}\bar{\varphi}_{m,m-e_j}(z)\psi_{m,m-e_j}(z+\zeta) & \text{if } [m]-[l]=e_j \\ 0 & \text{if } (m,l) \text{ not admissible}\end{cases} \end{aligned}$$

where $\psi_{m,m-e_j}(z)=(I_\alpha)^{-1}\varphi_{m,m-e_j}(z)$.

Consequently

$$n[U_\alpha]_{ml}=\begin{cases}\sum_i \dfrac{B_{ij}(z)}{\operatorname{Det} A(z)}\times \psi^{(l)}_{m,m-e_j}(z) & [m]-[l]=e_j\\ 0 & (m,\ l)\ \text{not admissible}\end{cases}$$

$$n[U_\alpha]_{m,m-e_j}=\frac{\operatorname{Det}\begin{bmatrix}\theta^0(nz_0) & \dots & \psi^0(nz_j) & \dots & \theta^0(nz_{n-1})\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \theta^{n-1}(nz_0) & \dots & \psi^0(nz_j) & \dots & \theta^{n-1}(nz_{n-1})\end{bmatrix}}{\operatorname{Det}\begin{bmatrix}\theta^0(nz_0) & \dots & \theta^0(nz_j) & \dots & \theta^0(nz_{n-1})\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \theta^{n-1}(nz_0) & \dots & \theta^0(nz_j) & \dots & \theta^{n-1}(nz_{n-1})\end{bmatrix}} \tag{22}$$

From (13), it can be shown that

$$\begin{aligned}
g^{-1}\varphi_{m,m-e_j}(z)&=(-1)\varphi_{m,m-e_j}(z+1)\\
h^{-1}\varphi_{m,m-e_j}(z)&=\exp\left[\frac{2\pi\mathrm{i}}{n}\left(z+n\bar{m}_jw+\frac{1}{2}+\frac{\tau}{2}\right)\right]\varphi_{m,m-e_j}(z+\tau)\\
\psi(z)=(I_\alpha)^{-1}\varphi_{m,m-e_j}(z)&=h^{-\alpha_2}g^{-\alpha_1}\varphi_{m,m-e_j}(z+\tau)\\
&=\exp\left[\frac{2\pi\mathrm{i}}{n}\left(\alpha_2 z+n\,\frac{\alpha_1}{2}+\alpha_1\alpha_2+\frac{\alpha_2^2\tau}{2}+n\alpha_2\bar{m}_jw+\frac{1}{2}\right)\right]\\
&\quad\times\varphi_{m,m-e_j}(z+\alpha_1+\alpha_2\tau)
\end{aligned} \tag{23}$$

$$\begin{aligned}
n[U^\alpha]_{ml}&=\bar{\varphi}_{mm-e_j}(z)(I_\alpha)^{-1}\varphi_{mm-e^j}(z)\\
&=\operatorname{Det}\begin{bmatrix}\theta^0(nz_0) & \dots & \theta^0(nz_j') & \dots & \theta^0(nz_{n-1})\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \theta^{n-1}(nz_0) & \dots & \theta^{n-1}(nz_j') & \dots & \theta^{n-1}(nz_{n-1})\end{bmatrix}\\
&\quad\times\operatorname{Det}\begin{bmatrix}\theta^0(nz_0) & \dots & \theta^0(nz_j) & \dots & \theta^0(nz_{n-1})\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \theta^{n-1}(nz_0) & \dots & \theta^{n-1}(nz_j) & \dots & \theta^{n-1}(nz_{n-1})\end{bmatrix}^{-1}\times\text{Factor}
\end{aligned} \tag{24}$$

where

$$z_i=\frac{1}{n}\,(z+nm_iw)\qquad z_j'=\frac{1}{n}\,(\alpha_1+\alpha_2\tau+\zeta)+z_j\qquad i,j=0,\ \dots,\ n-1 \tag{25}$$

$$\text{Factor}=\exp\left\{\frac{2\pi\mathrm{i}}{n}\left[\alpha_2(z+\zeta)+n\,\frac{\alpha_1}{2}+\alpha_1\alpha_2+\frac{\alpha_2^2\tau}{2}+n\alpha_2\bar{m}_jw+\frac{\alpha_2}{2}\right]\right\}.$$

Let

$$\mathrm{Det}\begin{bmatrix}\theta^0(nz_0) & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \theta^{n-1}(nz_{n-1})\end{bmatrix}=f(z_0). \tag{26}$$

As

$$\begin{aligned}\theta\begin{bmatrix}a\\b\end{bmatrix}(z+\tau,\ \tau)&=\exp(-\pi i\tau-2\pi i(z+b))\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\ \tau)\\ \theta\begin{bmatrix}a\\b\end{bmatrix}(z+1,\ \tau)&=\exp(2\pi ia)\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\ \tau)\end{aligned} \tag{27}$$

$$\begin{aligned}f(z_0+\tau)&=\exp(-\pi i\tau-2\pi i(nz_0+\tfrac{1}{2}))f(z_0)\\ f(z_0+1)&=\exp(n\pi ia)f(z_0).\end{aligned} \tag{28}$$

Thus $f(z_0)$ has n zeros in $\wedge_\tau$, and the sum of the zeros is

$$\sum_{\wedge_\tau}\text{zeros}=\frac{n-1}{2}.$$

As a function of z_0, $f(z_0)$ has $n-1$ obvious zeros, $z_0=z_1, z_2, \ldots, z_{n-1}$, so the last zero is at $z_0=-z_1-z_2-\ldots-z_{n-1}$. It is easy to check that the function

$$g(z_0)=\sigma_0\left(z_0+z_1+\ldots+z_{n-1}-\frac{n-1}{2}\right)\prod_{k\neq 0}\sigma_0(z_0-z_k)$$

has the same zeros as $f(z_0)$. As the function $f(z_0)/g(z_0)$ is a double periodic pure function, it can only be a z_0-independent constant, i.e.

$$\frac{f(z_0)}{g(z_0)}=C.$$

The same analysis is also true for z_i, $i=1, \ldots, n-1$, so we finally have

$$f(z_0)=\sigma_0\left(\sum_{i=0}^{n-1}z_i-\frac{n-1}{2}\right)\prod_{i<k}\sigma_0(z_i-z_k). \tag{29}$$

Applying this to (24) we have

$$n[U_\alpha]_{m,m-e_j}=\frac{\sigma_0\left(\sum_{i\neq j}z_i+z_j'-\frac{n-1}{2}\right)\prod_{k\neq j}\sigma_0(z_j'-z_k)}{\sigma_0\left(\sum_{i=0}^{n-1}z_i-\frac{n-1}{2}\right)\prod_{k\neq j}\sigma_0(z_j-z_k)}\times\text{Factor}$$

where z_i, z'_j are given by (25).

$$n[U_\alpha]_{m,m-e_j}=\frac{\sigma_0\left(z+w\delta+\dfrac{\alpha_1+\alpha_2\tau+\zeta}{n}-\dfrac{n-1}{2}\right)}{\sigma_0\left(z+\delta-\dfrac{n-1}{2}\right)}$$

$$\times\prod_{k\neq j}\frac{\sigma_0\left[\dfrac{\alpha_1+\alpha_2\tau+\zeta}{n}+(\bar{m}_j-\bar{m}_k)w\right]}{\sigma_0[(\bar{m}_j-\bar{m}_k)w]}\times\text{Factor.}$$

$$\delta=\sum_j\bar{m}_j=\sum_j w_j.$$

As

$$\sigma_0\left(z+\frac{\alpha_1+\alpha_2\tau}{n}\right)=\exp\left[\frac{-\pi\mathrm{i}}{n^2}\,(2n\alpha_2 z+2\alpha_1\alpha_2+\alpha_2^2\tau+n\alpha_2)\right]\times\sigma_\alpha(z)$$

so

$$[U_\alpha]_{m,m-e_j}=\frac{\sigma_\alpha\left(z+\dfrac{\zeta}{n}+w\delta-\dfrac{n-1}{2}\right)}{n\sigma_0\left(z+w\delta-\dfrac{n-1}{2}\right)}\prod_{k\neq j}\frac{\sigma_\alpha\left[\dfrac{\zeta}{n}+(\bar{m}_j-\bar{m}_k)w\right]}{\sigma_0[(\bar{m}_j-\bar{m}_k)w]}$$

$$\times\exp\left(-\frac{2\pi\mathrm{i}}{n}\left[\alpha_2\left(z+\frac{\alpha_2\tau+2\alpha_1+n}{2}+\xi+w\delta-\frac{n-1}{2}+n\bar{m}_jw-\sum\bar{m}_kw\right)\right]\right)$$

$$\times\exp\left(\frac{2\pi\mathrm{i}}{n}\left[\alpha_2(z+\xi)+\alpha_2^2\tau+\alpha_1\alpha_2+\frac{n\alpha_1}{2}+n\alpha_2\bar{m}_jw+\frac{\alpha_2}{2}\right]\right)$$

$$=\mathrm{e}^{\pi\mathrm{i}\alpha_1}\frac{\sigma_\alpha\left(z+\dfrac{\zeta}{n}+w\delta-\dfrac{n-1}{2}\right)}{n\sigma_0\left(z+w\delta-\dfrac{n-1}{2}\right)}\prod_{k\neq 1}\frac{\sigma_\alpha\left[\dfrac{\zeta}{n}+(m_j+w_j-m_kw_k)w\right]}{\sigma_0[(m_j+w_j-m_k-w_k)w]}. \tag{30}$$

Comparing this with (2) and (5) we can see that, aside from some insignificant terms such as $[n\sigma_0(z+w\delta-(n-1)/2)]^{-1}$, $L(z)$ in (16*b*) has the desired form of (6); we can consequently write the matrix element of the cyclic representation of the SA

$$(S_\alpha)_{m,m-e_j}-(-1)^{\alpha_1}\sigma_\alpha\left(\frac{w}{n}\right)\prod_{k\neq j}\frac{\sigma_\alpha\left[\dfrac{\zeta}{n}+(m_j+w_j-m_k-w_k)w\right]}{\sigma_0[(m_j+w_j-m_k-w_k)w]} \tag{31}$$

$$L(z)=\frac{\sum_\alpha W_\alpha\left(z-\dfrac{w}{n}+\dfrac{\zeta}{n}+w\delta-\dfrac{n-1}{2}\right)I_\alpha S_\alpha}{n\sigma_0\left(z+w\delta-\dfrac{n-1}{2}\right)}. \tag{32}$$

Equation (32) is invariant under $\alpha_i \to \alpha_i + n$ because

$$\sigma_{\alpha+(nq,np)}(z,\tau) = (-1)^q \exp(2\pi i q\alpha_2/n)\sigma_\alpha(z,\tau) \qquad p, q \in Z$$

$$S_\alpha \to S_\alpha \qquad \text{as } \alpha_2 \to \alpha_2 + n$$

$$S_\alpha \to (-1)^n \times \exp\left(2\pi i\left(\frac{1}{2} + \frac{\alpha_2}{n}\right) \times n\right) = S_\alpha \qquad \text{as } \alpha_1 \to \alpha_1 \to n.$$

Remark. S_α, $L(z)$ are operators acting on $W^{(0)}$ (see (7)), and their matrix elements are well defined on the standard bases w_m given in (7*a*) (i.e. they are invariant under $m_i \to m_i + N$, $m \succ (1, 1, \ldots, 1) + m$).

2.2. *Cocycle coefficient of the cyclic representation*

We can expand the SA in the following way:

As Sklyanin algebra (6) is defined by second-order homogeneous equations with respect to S_α, and the sum of the lower indices of two Ss are the same (mod $n \times n$). There are only n states $|m - e_i\rangle$, $i = 0, 1, \ldots, n-1$, from which we can get to $|m\rangle$ by acting S_α on them, and there are n^2 states $|m - e_i - e[k]\rangle$, $i, k = 0, 1, \ldots, n-1$, from which we can get to $|m\rangle$ by acting $S_\alpha S_\beta$ on them. The possible routes from $|m\rangle$ to $|m - e_i - e_k\rangle$ are

(a) $i = k$

$$|m - e_i - e_i\rangle \to |m - e_i\rangle \to |m\rangle$$

(b) $i \neq k$

(i) $|m_i - e_k\rangle \to |m - e_i\rangle \to |m\rangle$

(ii) $|m - e_i - e_k\rangle \to |m - e_k\rangle \to |m\rangle$.

We propose to expand S_α as

$$(S_\alpha)_{m,m-e_j} \to e^{A(m,j,\alpha)}(S_\alpha)_{m,m-e_j}. \tag{33}$$

If

$$A(m, i, \alpha - \gamma) + A(m - e_i, k, \gamma) = B(m, i, k, \alpha) \tag{34}$$

is independent of $\gamma, m - e_i$, then (6) is still true, we require $e^{A(m,j,\alpha)}$ to be well defined with respect to m, $\alpha \in Z_n^2$, the simplest non-trivial choice is

$$A(m, j, \alpha) = \delta_j + \frac{2\pi i}{n}(l_1\alpha_1 + l_2\alpha_2)$$

where $\delta_j \in C$, $l_1, l_2 \in Z$.

The second term in our choice of $A(m, j, \alpha)$ is actually the part isomorphic to Sklyanin algebra. Finally we have

$$(S_\alpha)_{m,m-e_j} = \frac{\exp\left(\delta_j + 2\pi i\left(\frac{\alpha_1}{2} + l_1\frac{\alpha_1}{n} + l_2\frac{\alpha_2}{n}\right)\right)}{\sigma_\alpha\left(\frac{w}{n}\right)} \prod_{k \neq j} \frac{\sigma_\alpha\left[\frac{\zeta}{n} + (m_j + w_j - m_k - w_k)w\right]}{\sigma_0[(m_j + w_j - m_k - w_k)w]} \tag{35}$$

in which we have the parameters $\delta_j, \zeta, w_j - w_k$. As, when $\delta_j \to \delta_j + v$, S'_α is essentially unchanged, we choose $\sum \delta_j = 0$, so we have $2n-1$ independent parameters in the representation.

It is remarkable that we cannot get rid of the coefficient

$$\exp\left(\delta_j + 2\pi\mathrm{i}\left(\frac{\alpha_1}{2} + l_1\frac{\alpha_1}{n} + l_2\frac{\alpha_2}{n}\right)\right)$$

through a similar transformation, because as we take the sequence $|m - Ne_i\rangle \to \ldots \to |m\rangle$, the product of these coefficients does not give one. As the initial and the final states in the sequence are identical, similar transformation should give a trivial coefficient.

$$\exp\left(\delta_j + 2\pi\mathrm{i}\left(\frac{\alpha_1}{2} + l_1\frac{\alpha_1}{n} + l_2\frac{\alpha_2}{n}\right)\right)$$

is actually a cocycle coefficient on n-torus.

2.3. *Function difference representation of* SA

So far in our treatment we have restricted ourselves to the case where $w = N'/N$ is rational, and we find that the representation is well defined on $\{Q\}$. We can ease our restrictions such that we associate each bases vector with (m), $m = \{m_0, \ldots, m_{n-1}\} \in Z^n$. The admissible condition is still $(m) - (1) = (e_j)$, i.e. $m_i - l_i = \delta_{ij}$. And we take (35) as $(\bar{S}_\alpha)_{(m)(m-e_j)}$. Obviously we have

$$\sum_{\gamma m'} C_{\alpha\beta\gamma}(\bar{S}_{\alpha-\gamma})_{(m)(m')}(\bar{S}_\gamma)_{(m')(m'')} = 0. \tag{36}$$

According to Jimbo *et al* [8], equation (36) holds irrespective of whether w is a rational number or not.

Let $(m_i - w_i)w \equiv u_i$, then (35) becomes

$$(S_\alpha)_{m,m-e_j}(w) \equiv (S_\alpha)_j(u, w)$$

$$= \exp\left(\delta_j + 2\pi\mathrm{i}\left(\frac{\alpha_1}{2} + l_1\frac{\alpha_1}{n} + l_2\frac{\alpha_2}{n}\right)\right) \times \sigma_\alpha\left(\frac{w}{n}\right) \times \prod_{k\neq j} \frac{\sigma_\alpha\left[\frac{\zeta}{n} + (u_j - u_k)w\right]}{\sigma_0[(u_j - u_k)w]}. \tag{37}$$

Then (6) becomes

$$\text{(a)}\ \ m'' = m - 2e_i \qquad \sum_\gamma C_{\alpha\beta\gamma}(S_{\alpha-\gamma})_i(u)(S_\gamma)_i(u_0, \ldots, u_i - w, \ldots) = 0$$

$$\text{(b)}\ \ m'' = m - e_j - e_k \qquad j \neq k;$$

$$\sum_\gamma C_{\alpha\beta\gamma}\{(S_{\alpha-\gamma})_j(u)(S_\gamma)_k(u_0, \ldots, u_j - w, \ldots) \tag{38}$$

$$+ (S_{\alpha-\gamma})_k(u)(S_\gamma)_j(u_0, \ldots, u_k - w, \ldots)\} = 0.$$

As w_j is generic, so is $u \equiv \{u_j\}$, and it is easy to see that (38) is true for generic u.

Next we consider the function difference representation of SA. Define the operator $\hat{S}_\alpha$ acting on function $f(u) = f(u_0, \ldots, u_{n-1})$ as

$$\hat{S}_\alpha f(u) = \sum_j (S_\alpha)_j(u, w) f(u_0, \ldots, u_{n-1})$$

then

$$\sum_\gamma C_{\alpha\beta\gamma}\hat{S}_{\alpha-\gamma}\hat{S}_\gamma f(u)$$
$$=\sum_\gamma C_{\alpha\beta\gamma}\hat{S}_{\alpha-\gamma}\sum_k (S_\gamma)_k f(u_0, \ldots, u_k-w, \ldots)$$
$$=\sum_\gamma C_{\alpha\beta\gamma}\sum_j (S_{\alpha-\gamma})_j(u)$$
$$\times\Bigg[\sum_{k\neq j}(S_\gamma)_k(u_0, \ldots, u_j-w, \ldots)$$
$$\times f(u_0, \ldots, u_j-w, \ldots, u_k-w, \ldots)$$
$$+(S_\gamma)_j(u_0, \ldots, u_j-w, \ldots)f(u_0, \ldots, u_j-2w, \ldots)\Bigg]$$
$$=\sum_\gamma C_{\alpha\beta\gamma}\sum_{j<k}[(S_{\alpha-\gamma})_j(u)(S_\gamma)_k(u_0, \ldots, u_j-w, \ldots)$$
$$+(S_{\alpha-\gamma})_k(u)(S_\gamma)_j(u_0, \ldots, u_k-w, \ldots)]$$
$$\times f(u_0, \ldots, u_j-w, \ldots, u_k-w, \ldots)$$
$$+\sum_\gamma C_{\alpha\beta\gamma}\sum_j (S_{\alpha-\gamma})_j(u)(S_\gamma)_j(u_0, \ldots, u_j-w, \ldots)$$
$$\times f(u_0, \ldots, u_j-2w, \ldots).$$

So (39) guarantees the identity

$$\sum_\gamma C_{\alpha\beta\gamma}\hat{S}_{\alpha-\gamma}\hat{S}_\gamma f(u)=0.$$

This is the function difference representation of SA.

We now compare our results with those given by Sklyanin at $n=2$. At $n=2$, we consider the function as $f(u_0, u_1)=f(2u)$, where $u_0-u_1=2u$, $u_0+u_1=2v$. Let $\delta_0=\delta_1=\pi i$, $l_0=0$, $l_2=1$ in (37). We then have

$$(S_\alpha)_0(u)=\sigma_\alpha\left(\frac{w}{2}\right)\frac{\sigma_\alpha\left(\frac{\zeta}{n}+2u\right)}{\sigma_0(2u)}-(-1)^{\alpha_1+\alpha_2-1}$$

$$(S_\alpha)_1(u)=\sigma_\alpha\left(\frac{w}{2}\right)\frac{\sigma_\alpha\left(\frac{\zeta}{n}-2u\right)}{\sigma_0(-2u)}(-1)^{\alpha_1+\alpha_2-1}.$$

So

$$S_\alpha f(2u)=\left[\frac{\sigma_\alpha\left(\frac{w}{2}\right)\sigma_\alpha\left(\frac{\zeta}{n}+2u\right)}{\sigma_0(2u)}f(2u-w)-\frac{\sigma_\alpha\left(\frac{w}{2}\right)\sigma_\alpha\left(\frac{\zeta}{n}-2u\right)}{\sigma_0(2u)}f(2u+w)\right]\times(-1)^{\alpha_1+\alpha_2-1}$$

where $f(2u)=F(u)$, $\zeta/n=lw$.

$$S_\alpha F(u)=\left[\frac{\sigma_\alpha\left(\frac{w}{2}\right)\sigma_\alpha(lw+2u)}{\sigma_0(2u)}F\left(u-\frac{w}{2}\right)-\frac{\sigma_\alpha\left(\frac{w}{2}\right)\sigma_\alpha(lw-2u)}{\sigma_0(2u)}F\left(u+\frac{w}{2}\right)\right]\times(-1)^{\alpha_1+\alpha_2-1}$$

$$\equiv S_\alpha\left(u+\frac{lw}{2}\right)F\left(u-\frac{w}{2}\right)-S_\alpha\left(-u+\frac{lw}{2}\right)F\left(u+\frac{w}{2}\right)$$

where $f(2u)=F(u)$, $\zeta/n=lw$.

Let $w/2=-\eta$, remember that $\sigma_{00}(-\eta)=0_{00}(-\eta)=-\sigma_{00}(\eta)$, $\sigma_i(-\eta)=\sigma_i(\eta)$, $i=\alpha_2, \alpha_1=1,0;\ 1,1;\ 01$. We have

$$S_\alpha F(u)=\frac{S_\alpha(u-l\eta)F(u+\eta)-S_\alpha(-u-l\eta)F(u-n)}{0_{11}(2u)}$$

where S_α and the corresponding I_α, S_α are

$$S_\alpha=\begin{cases}S_{00}=\theta_{11}(\eta)\theta_{11}(2u)\\ S_{01}=\theta_{10}(\eta)\theta_{10}(2u)\\ S_{10}=\theta_{01}(\eta)\theta_{01}(2u)\\ S_{11}=-\theta_{00}(\eta)\theta_{00}(2u)\end{cases}\qquad I_\alpha=\begin{cases}I=I_{00}\\ \sigma_z=I_{01}\\ \sigma_x=I_{10}\\ i\sigma_y=I_{11}\end{cases}\qquad S_\alpha=\begin{cases}S_0\\ S_3\\ S_1\\ S_2\end{cases}$$

where σ_x, σ_y, σ_z are the Pauli matrices, and S_0, S_1, S_2, S_3 are the corresponding Sklyanin operators. From the automorphism of Sklyanin $(S_0, S_1, S_2, S_3)\to(S_0, S_3, -S_2, S_1)$, we have

$$S_0'=\theta_{11}(\eta)\theta_{11}(2u)\qquad S_1'=\theta_{10}(\eta)\theta_{10}(2u)$$

$$S_2'=\theta_{00}(\eta)\theta_{00}(2u)\qquad S_3'=\theta_{01}(\eta)\theta_{01}(2u).$$

Comparing S_α' with S_α'' used by Sklyanin

$$L(z)=\sum W_\alpha(z)\sigma_\alpha S_\alpha''$$

we see that $iS_2'=S_2''$, $S_j'=S_j''$, $j=0,1,3$.

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